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2°. The integral under consideration is known as the Beta function, or first Eulerian integral, and is denoted by $B(m, n)$. This function is to be expressed in terms of the Gamma function, or second Eulerian integral, $\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx$. The required relation is

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

In

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{1}$$

set $x = y/(1+y)$, obtaining,

$$B(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy. \tag{2}$$

In

$$\Gamma(m) = \int_0^\infty e^{-z}z^{m-1}dz,$$

if we set $z = ax$, we get the formula,

$$\frac{1}{a^m} = \frac{1}{\Gamma(m)}\int_0^\infty e^{-ax}x^{m-1}dx. \tag{3}$$

In (3) replace a by $1 + y$ and m by $m + n$, obtaining,

$$\frac{1}{(1+y)^{m+n}} = \frac{1}{\Gamma(m+n)}\int_0^\infty e^{-(1+y)x}x^{m+n-1}dx. \tag{4}$$

(2) and (4) give

$$B(m, n) = \frac{1}{\Gamma(m+n)}\int_0^\infty dy\int_0^\infty e^{-(1+y)x}x^{m+n-1}y^{m-1}dx. \tag{5}$$

In (5) the order of integration may be inverted.[2] For, (1) the integrand, $f(x, y)$, is continuous in the first quadrant, except on the axes; (2) the integrals, $\int_0^\infty f(x, y)dx$ and $\int_0^\infty f(x, y)dy$ are uniformly convergent[3] in any intervals (a, b), $a > 0$, and (α, β), $\alpha > 0$, respectively; and (3) $\int_0^\infty dx\int_0^\infty f(x, y)dy$ exists, since

$$\begin{aligned}\int_0^\infty f(x, y)dy &= e^{-x}x^{m+n-1}\int_0^\infty e^{-xy}y^{m-1}dy\\ &= e^{-x}x^{m+n-1}\cdot x^{-m}\Gamma(m), \text{ by (3)}\\ &= e^{-x}x^{n-1}\Gamma(m)\end{aligned} \tag{6}$$

and, therefore, for $a > 0$, using (6),

$$\int_0^\infty dx\int_0^\infty f(x, y)dy = \Gamma(m)\lim_{a=0}\int_a^\infty e^{-x}x^{n-1}dx = \Gamma(m)\Gamma(n). \tag{7}$$

The conditions for inversion being fulfilled, (5) and (7) give the desired result,

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

383. Proposed by WILLIAM CULLUM, Albion, Mich.

Find the area of the curved surface of a right cone whose base is the asteroid, $x^{2/3} + y^{2/3} = a^{2/3}$, and whose altitude is h. From Townsend and Goodenough's *First Course in Calculus*, p. 288, Ex. 11.

Note. Among other methods, find the required area by means of the formula

$$\int\int\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dydx.$$

EDITORS.

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[2] See Nielson's *Handbuch d. Theorie d. Gamma Functionen*, pages 148–9. Pierpont, *loc. cit.*, § 680, 3.

[3] Pierpont, *loc. cit.*, § 663.

TWO SOLUTIONS BY F. L. GRIFFIN, Reed College.

(I) *By a single integration.* The element of area is a triangle, whose base ds is an elementary arc of the asteroid, and whose altitude H is the perpendicular from the vertex of the cone upon the tangent to the arc ds. This perpendicular is the hypotenuse of a right triangle whose legs are the height of the cone h, and the perpendicular p from the center of the base upon the tangent. Hence

$$dS = \tfrac{1}{2}H\,ds, \qquad \text{where} \qquad H = \sqrt{h^2 + p^2}.$$

From the parametric equations of the asteroid $x = a\cos^3 t$, $y = a\sin^3 t$, we find $ds = \sqrt{dx^2 + dy^2} = \tfrac{3}{2}a\sin 2t\,dt$. Also, since $dy/dx = -\tan t$, the equation of the tangent line is $Y - a\sin^3 t = -\tan t(X - a\cos^3 t)$. On reducing this to the normal form we find

$$p = \frac{a\sin^3 t + \tan t \cdot a\cos^3 t}{\sec t} = \tfrac{1}{2}a\sin 2t;$$

whence

$$H = \sqrt{h^2 + \frac{a^2}{4}\sin^2 2t} = \tfrac{1}{2}\sqrt{(4h^2 + a^2) - a^2\cos^2 2t}.$$

Finally,

$$S = \frac{3a}{2}\int_0^{\pi/2} \sqrt{(4h^2 + a^2) - a^2\cos^2 2t} \cdot \sin 2t\,dt,$$

which reduces, on putting $\cos 2t = z$, to

$$S = \tfrac{3}{4}a^2\int_{-1}^{1}\sqrt{\frac{4h^2 + a^2}{a^2} - z^2}\,dz = \tfrac{3}{2}ah + \tfrac{3}{4}(4h^2 + a^2)\sin^{-1}\frac{a}{\sqrt{4h^2 + a^2}}.$$

[This result, by the way, has a simple interpretation: Three times the right triangle which forms half of the broadest vertical section of the cone, increased by three times a circular sector whose radius is the smallest slant height of the cone and whose angle is the angle between two such opposite smallest slant heights.]

(II) *By the formula* $\int\int\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\,dy\,dx$. The equation of the surface, taking the origin at the center of the base, is

$$z = h - \frac{h}{a}(x^{2/3} + y^{2/3}),^{3/2}$$

whence

$$S = 4\int_0^a\int_0^{(a^{2/3}-x^{2/3})^{3/2}}\sqrt{x^{2/3}y^{2/3} + \frac{h^2}{a^2}(x^{2/3} + y^{2/3})^2}\,x^{-1/3}y^{-1/3}\,dy\,dx.$$

Put $x = au^{3/2}$, $y = av^{3/2}$, $h/a = 1/k$, and we have

$$S = \frac{9a^2}{k}\int_0^1\int_0^{1-u}\sqrt{k^2uv + (u + v)^2}\,dv\,du$$

$$= \frac{9a^2}{k}\int_0^1 du\left[\frac{2v + (2 + k^2)u}{8}\sqrt{(2v + \overline{2 + k^2}u)^2 - (k^4 + 4k^2)u^2}\right.$$
$$\left. - \frac{(k^4 + 4k^2)u^2}{8}\log\left(\frac{2v + \overline{2 + k^2}u + \sqrt{(2v + \overline{2 + k^2}u)^2 - (k^4 + 4k^2)u^2}}{2}\right)\right]_0^{1-u}$$

$$= I_1 - I_2 - I_3 + I_4,$$

where

$$I_1 = \frac{9a^2}{4k}\int_0^1 (2 + k^2u)\sqrt{1 + k^2u - k^2u^2}\,du, \qquad I_2 = \frac{9a^2(k^2 + 2)}{4k}\int_0^1 u^2du = \frac{3a^2(k^2 + 2)}{4k},$$

$$I_3 = \frac{9a^2k(k^2 + 4)}{8}\int_0^1 u^2\log(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2})\,du,$$

$$I_4 = \frac{9a^2k(k^2 + 4)}{8}\int_0^1 u^2[\log u + \log(k^2 + 4)]\,du = \frac{a^2k(k^2 + 4)}{8}[\log(k^2 + 4)^3 - 1].$$

To evaluate I_1, complete the square and put

$$u = \frac{1}{2}\left(1 + \frac{\sqrt{k^2+4}}{k}\sin\theta\right).$$

Then

$$\sqrt{1 + k^2u - k^2u^2} = \frac{\sqrt{k^2+4}}{2}\cos\theta,$$

and $2 + k^2u = \frac{1}{2}(k^2+4)(1 + \sin\alpha\sin\theta)$ where $\alpha = \sin^{-1}(k/\sqrt{k^2+4}) = \cos^{-1}(2/\sqrt{k^2+4})$. Hence,

$$I_1 = \frac{9a^2(k^2+4)^2}{32k^2}\int_{-\alpha}^{\alpha}(1 + \sin\alpha\sin\theta)\cos^2\theta d\theta$$

$$= \frac{9a^2(k^2+4)^2}{32k^2}[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{1}{3}\sin\alpha\cos^3\theta]_{-\alpha}^{\alpha} = \frac{9a^2(k^2+4)^2}{32k^2}\left[\alpha + \frac{2k}{k^2+4}\right].$$

To evalute I_3, first integrate by parts, putting

$$U = \log(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2}), \qquad dV = \frac{9a^2k(k^2+4)}{8}u^2du.$$

Then

$$dU = \frac{k^2du}{2R}\left\{1 - \frac{(k^2+4)u}{2 + k^2u + 2R}\right\}, \qquad V = \frac{3a^2k(k^2+4)}{8}u^3,$$

where $R = \sqrt{1 + k^2u - k^2u^2}$. Then

$$I_3 = \frac{3a^2k(k^2+4)}{8}\left\{[u^3\log(2 + k^2u + 2R)]_0^1 - \frac{k^2}{2}\int_0^1\frac{u^3du}{R} + \frac{k^2(k^2+4)}{2}\int_0^1\frac{u^4du}{R(2 + k^2u + 2R)}\right\}$$

$$= \frac{3a^2k(k^2+4)}{8}\log(k^2+4) - I_5 + I_6,$$

where

$$I_5 = \frac{3a^2k^3(k^2+4)}{16}\int_0^1\frac{u^3du}{\sqrt{1 + k^2u - k^2u^2}} = \frac{3a^2(k^2+4)^{\frac{5}{2}}}{128k}\int_{-\alpha}^{\alpha}(\sin\alpha + \sin\theta)^3d\theta,$$

$$= \frac{3a^2(k^2+4)^{\frac{5}{2}}}{128k}[2\alpha\sin^3\alpha + 3\alpha\sin\alpha - 3\sin^2\alpha\cos\alpha],$$

$$= \frac{3a^2(k^2+4)}{128}[(5k^2+12)\alpha - 6k],$$

and

$$I_6 = \frac{3a^2k^3(k^2+4)^2}{16}\int_0^1\frac{u^4du}{\sqrt{1 + k^2u - k^2u^2}(2 + k^2u + 2\sqrt{1 + k^2u - k^2u^2})}$$

$$= \frac{3a^2(k^2+4)^3}{128k^2}\int_{-\alpha}^{\alpha}\frac{(\sin\alpha + \sin\theta)^4d\theta}{\sin\alpha\sin\theta + \cos\alpha\cos\theta + 1},$$

since

$$2 + k^2u = \tfrac{1}{2}(k^2+4)(1 + \sin\alpha\sin\theta) \qquad \text{and} \qquad 2R = \sqrt{k^2+4}\cos\theta = \tfrac{1}{2}(k^2+4)\cos\alpha\cos\theta.$$

Hence,

$$I_6 = \frac{3a^2(k^2+4)^3}{128k^2}\int_{-\alpha}^{\alpha}\frac{\left(2\sin\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}\right)^4d\theta}{2\cos^2\frac{\theta-\alpha}{2}}$$

$$= \frac{3a^2(k^2+4)^3}{16k^2}\int_{-\alpha}^{\alpha}\sin^4\left(\frac{\theta+\alpha}{2}\right)\cos^2\frac{\theta-\alpha}{2}d\theta.$$

By making imaginary exponential substitutions, or putting $\theta = 2\phi + \alpha$ and expanding, we find

$$I_6 = \frac{3a^2(k^2+4)^3}{512\mathrm{k}^2}\left[\sin(2\theta+2\alpha) - 8\sin(\theta+\alpha) + 6\theta + \tfrac{1}{3}\sin(3\theta+\alpha) + \sin(\theta+3\alpha) - 2\sin 2\theta - 4\theta\cos 2\alpha + 6\sin(\theta-\alpha)\right]_{-\alpha}^{\alpha}$$

$$= \frac{3a^2(k^2+4)^3}{128k^2}\left[\tfrac{14}{3}\sin\alpha\cos^3\alpha - \tfrac{17}{3}\sin\alpha\cos\alpha + \alpha(1+4\sin^2\alpha)\right]$$

$$= \frac{a^2(k^2+4)}{128k^2}\left[-2k(17k^2+12) + (15k^2+12)(k^2+4)\alpha\right].$$

Collecting results we find, since $S = I_1 - I_2 - I_3 + I_4$,

$$S = \alpha\left\{\frac{9a^2(k^2+4)^2}{32k^2} + \frac{3a^2(k^2+4)(5k^2+12)}{128} - \frac{a^2(k^2+4)^2(15k^2+12)}{128k^2}\right\}$$

$$+ \frac{a^2}{64k}\{36(k^2+4) - 48(k^2+2) - 8k^2(k^2+4) - 9k^2(k^2+4) + (17k^2+12)(k^2+4)\}$$

$$= \frac{3}{2}\frac{a^2}{k} + \tfrac{3}{4}a^2\frac{k^2+4}{k^2}\alpha = \tfrac{3}{2}ah + \tfrac{3}{4}(4h^2+a^2)\sin^{-1}\frac{a}{\sqrt{4h^2+a^2}}.$$

Also solved by G. W. HARTWELL and the PROPOSER.

398A. Proposed by EMMA GIBSON, Drury College.

[This problem was unnumbered in the December issue.]

Solve the differential equation

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)x\frac{dy}{dx} = 0.$$

SOLUTION BY FREDERICK WOOD, University of Wisconsin.

The equation

$$(x^3y^3 + x^2y^2 + xy + 1)y + (x^3y^3 - x^2y^2 - xy + 1)x\frac{dy}{dx} = 0$$

reduces to

$$(x^3y^3 + 1)(ydx + xdy) + (x^2y^2 + xy)(ydx - xdy) = 0$$

and finally to

$$x^2y^2(ydx + xdy) + ydx + xdy - 2x^2ydy = 0$$

from which we get

$$ydx + xdy + \frac{ydx + xdy}{x^2y^2} - \frac{2dy}{y} = 0.$$

This becomes, after integration,

$$xy - \frac{1}{xy} - 2\log y = c, \qquad \text{or} \qquad x^2y^2 - 2xy\log c'y = 1.$$

Also solved by H. POLISH, W. W. BEMAN, H. S. UHLER, ELIJAH SWIFT, HORACE OLSON, S. E. RASOR, G. KEULIGAN, H. L. AGARD, ELMER SCHUYLER, and B. J. BROWN.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

REPLIES.

24. The following facts are significant:

(1) The New England Association of Mathematics Teachers has appointed a committee "to investigate the current criticisms of high school mathematics."

(2) A committee of the Council of the American Mathematical Society has under consideration the question "whether any action is desirable on the part of the Society in the matter of the movement against mathematics in the schools."

(3) At the recent meeting in Cincinnati of the National Education Association an iconoclastic discussion on the topic: "Can algebra and geometry be reorganized so as to justify their